JOURNAL OF APPROXIMATION THEORY 20, 251–256 (1977)

The Limit of Transformed Rational L_p Approximation

CHARLES B. DUNHAM

Computer Science Department, University of Western Ontario, London, Ontario, Canada

Communicated by Philip J. Davis

Received March 17, 1975

A continuous function on [0, 1] is approximated by a family of functions of the form $\sigma(R(A, \cdot))$, where $R(A, \cdot)$ is a generalized rational function, with respect to an L_p norm. Sufficient conditions are given for a limit of best L_k parameters to be a best L_∞ parameter.

Let $C[0, 1]$ be the space of continuous functions on [0, 1]. For g a bounded measurable function on [0, 1] define

$$\|g\|_p = \left[\int_0^1 |g(x)|^p \, dx\right]^{1/p}, \qquad 1 \leqslant p < \infty$$

$$\|g\|_\infty = \sup\{|g(x)|: 0 \leqslant x \leqslant 1\}.$$

Let $\{\phi_1, ..., \phi_n\}$, $\{\psi_1, ..., \psi_m\}$ be linearly independent subsets of $C[0, 1]$ and define

$$R(A, x) = P(A, x)/Q(A, x) = \sum_{k=1}^{n} a_k \phi_k(x) \Big/ \sum_{k=1}^{m} a_{n+k} \psi_k(x).$$

Let σ be a continuous function from the real line into the extended real line and define

$$F(A, x) = \sigma(R(A, x)).$$

Let P be a subset of $(n + m)$-space. The L_p approximation problem is: Given $f \in C[0, 1]$, to find $A^* \in P$ for which $e_p(A) = \|f - F(A, \cdot)\|_p$ attains its infimum $\rho_p(f)$ over $A \in P$. Such a parameter A^* is called a best L_p parameter and $F(A^*, \cdot)$ is called a best L_p approximation to f.

This paper is concerned with the behavior of a sequence of best L_p parameters as $p \to \infty$. The only case in which this has previously been considered was that in which F is a linear approximating function [4, pp. 8–10] and that in which F is a unisolvent approximating function [6].

$F(A, x)$ is well defined if $Q(A, x) \neq 0$. When $Q(A, x) = 0$ we need a

ISSN 0021-9045

convention to define $F(A, x)$. We will assume in this paper that Q has the zero measure and dense nonzero properties, that is, if $Q(A, \cdot) \not\equiv 0$, then the set of zeros of $Q(A, \cdot)$ is of zero measure and the set of points at which $Q(A, \cdot)$ does not vanish is dense in [0, 1]. This is a combination of the hypothesis used in [2, 3] and the hypothesis of Boehm [1, 5, p. 84]. The zero measure property makes it possible for us to ignore the points where $Q(A, \cdot)$ vanishes when using the L_p norms. The dense nonzero property enables us to use Boehm's convention [1, p. 20] to define $F(A, x)$ where $Q(A, x) = 0$ (defining $F(A, x)$ there is only necessary with the L_∞ norm).

As $R(\alpha A, x) = R(A, x)$ for all $\alpha > 0$, we will normalize parameters A so that

$$\sum_{k=1}^{m} |a_{n+k}| = 1. \tag{1}$$

Let $\hat{P}$ be the set of parameters satisfying (1). It is proven in [2] that $R(A, \cdot)$ is measurable if $A \in \hat{P}$ and Q has the zero measure property. As σ is Borel measurable, $F(A, \cdot)$ is then measurable.

THEOREM 1. *Let Q have the zero measure property, P be a closed non-empty subset of $\hat{P}$, and $|\sigma(t)| \to \infty$ as $|t| \to \infty$, then a best L_p approximation exists to bounded measurable f.*

We use the arguments of [3] (without modification) to prove this. Reference [3] considers the (closed) set of parameters

$$P_0 = \{A : A \in \hat{P}, Q(A, \cdot) \geq 0\}.$$

In [3] are given several closed subsets of P_0.

LEMMA 1. *Let g be measurable, then $\|g\|_p \leq \|g\|_\infty$.*

Proof. $\|g\|_p \leq \|\sup\{|g(x)| : 0 \leq x \leq 1\}\|_p = \|g\|_\infty$.

THEOREM 2. *Let there exist $C \in P$ with $\|f - F(C, \cdot)\|_\infty < \infty$. Let Q have the zero measure and dense nonzero properties. Let P be a closed nonempty subset of $\hat{P}$. Let $|\sigma(t)| \to \infty$ as $|t| \to \infty$. Let $p(k) \to \infty$ and A_k be a best $L_{p(k)}$ parameter. Then $\{A^k\}$ has an accumulation point and any accumulation point is a best L_∞ parameter. Further $\{\rho_{p(k)}(f)\} \to \rho_\infty(f)$.*

Proof. Define the seminorm

$$\|A\| = \max\{|a_i| : 1 \leq i \leq n\}.$$

Suppose $\{\|A_k\|\}$ is unbounded, then by taking a subsequence if necessary we can assume $\{\|A_k\|\} \to \infty$. Define $B_k = A_k/\|A_k\|$, then $\|B_k\| = 1$. $\{B^k\}$ has an accumulation point B, $\|B\| = 1$, assume $\{B^k\} \to B$. There exists $x, \epsilon > 0$ such that $|P(B, x)| > \epsilon$. By continuity of $P(B, \cdot)$ there is a closed neighborhood N of x such that $|P(B, y)| > \epsilon$ for $y \in N$. There exists K such that $k \in K$ implies $|P(B_k, y)| > \epsilon$ for $y \in N$, hence

$$|P(A_k, y)| > \|A_k\| \epsilon, \qquad y \in N, \quad k > K,$$

$$|R(A_k, y)| > \|A_k\| \epsilon \Big/ \sum_{k=1}^{m} \|\psi_k\|_\infty, \qquad y \in N, \quad k > K.$$

and

$$m_k = \inf\{|R(A_k, y)| : y \in N\} \to \infty,$$
$$M_k = \inf\{|f(y) - \sigma(R(A_k, y))| : y \in N\} \geq \inf\{|\sigma(t)| : m_k \geq |t|\} - \|f\|_\infty \to \infty$$
$$\|f - \sigma(R(A, \cdot))\|_{p(k)} \geq M_k \mu(N)^{1/p(k)} \to \infty. \tag{2}$$

But by Lemma 1, $\|f - F(C, \cdot)\|_p \leq \|f - F(C, \cdot)\|_\infty < \infty$ and so (2) contradicts A_k being best with respect to $\| \|_{p(k)}$ for all k. Thus $\{\|A_k\|\}$ is bounded, that is, the numerator coefficients of $\{A^k\}$ are bounded. The denominator coefficients are bounded by the normalization (1). Thus $\{A^k\}$ is bounded and has an accumulation point A, assume that $\{A_k\} \to A$.

Next suppose that $e_\infty(A) = \infty$. Let $M > 0$ be given then there exists a point x with $Q(A, x) \neq 0$ such that $|f(x) - F(A, x)| > 2M$. By continuity of $F(A, \cdot)$ at x into the extended real line and continuity of $Q(A, \cdot)$ there is a closed neighborhood N of x such that $|f(y) - F(A, y)| > 2M$ for $y \in N$ and $Q(A, \cdot)$ does not vanish on N. There exists K such that for $k > K$,

$$|f(y) - F(A_k, y)| > M \qquad \text{for} \quad y \in N,$$
$$\|f - F(A_k, \cdot)\|_{p(k)} \geq \left[\int_N |f - F(A_k, \cdot)|^{p(k)}\right]^{1/p(k)} \geq 2M(\mu(N))^{1/p(k)} \to M.$$

As this would be true for all M, $\|f - F(A_k, \cdot)\|_{p(k)} \to \infty$. But we earlier showed this is impossible, hence $e_\infty(A) < \infty$,

We prove next that

$$\limsup_{k \to \infty} e_{p(k)}(A_k) \geq e_\infty(A). \tag{3}$$

Suppose not, then we must have for some $\epsilon > 0$ and all k,

$$e_{p(k)}(A_k) < e_\infty(A) - \epsilon. \tag{4}$$

By Boehm's convention there is a point x such that $|f(x) - F(A, x)| > e_\infty(A) - (\epsilon/4)$ and $Q(A, x) \neq 0$. By continuity of $f - F(A, \cdot)$ and $Q(A, \cdot)$ at x there is a closed neighborhood N of x such that

$$|f(y) - F(A, y)| \geq e_\infty(A) - (\epsilon/2), \qquad Q(A, y) \neq 0, y \in N.$$

As $R(A_k, \cdot)$ converges uniformly to $R(A, \cdot)$ on N, $F(A_k, \cdot)$ converges uniformly to $F(A, \cdot)$ on N and there exists K such that for $k > K$,

$$|f(y) - F(A_k, y)| \geq e_\infty(A) - (\epsilon/4) \qquad Q(A_k, y) > 0,\ y \in N.$$

We have

$$\|f - F(A_k, \cdot)\|_{p(k)} \geq \left[\int_N |f - F(A_k, \cdot)|^{p(k)}\right]^{1/p(k)}$$

$$\geq (e_\infty(A) - \epsilon/4)(\mu(N))^{1/p(k)} \to e_\infty(A) - (\epsilon/4),$$

contradicting (4) and proving (3). By Lemma 1,

$$e_{p(k)}(A^k) \leq e_{p(k)}(A) \leq e_\infty(A)$$

so we have in fact

$$\limsup_{k \to \infty} \rho_{p(k)}(f) = e_\infty(A).$$

Since this remains true for every subsequence of $\{p(k)\}$, we have

$$\lim_{k \to \infty} \rho_{p(k)}(f) = e_\infty(A).$$

Now suppose A is not best with respect to $\|\ \|_\infty$, then there is B, $\epsilon > 0$ with

$$\|f - F(B, \cdot)\|_\infty < \|f - F(A, \cdot)\|_\infty - \epsilon.$$

There is, therefore, k such that

$$\|f - F(B, \cdot)\|_{p(k)} \leq \|f - F(B, \cdot)\|_\infty < \|f - F(A_k, \cdot)\|_{p(k)},$$

contradicting optimality of A_k.

The theorem may not be true if we approximate by admissible approximations (ones with denominators greater than 0).

EXAMPLE. Let $F(A, x) = a_1 x/(a_2 + a_3 x)$ and $f = 1$. We have

$$x(k^{-1} + x)^{-1} \to 1 = f(x), \qquad x > 0,$$

hence $\rho_p(f) = 0$ for $1 \leq p < \infty$. Since $F(A, 0) = 0$ for all admissible A, 0 is a best L_∞ approximation and $\rho_\infty(f) = 1$.

COROLLARY. *Let the hypotheses of the previous theorem hold. Let $A_{k(j)} \to A$ then $F(A_{k(j)}, \cdot)$ converges pointwise to $F(A, \cdot)$ ourside the zeros of $Q(A, \cdot)$.*

If $\{A_k\} \to A$ and $Q(A, \cdot)$ has no zeros, $\{F(A_k, \cdot)\}$ converges uniformly to $F(A, \cdot)$, hence

THEOREM 3. *Let the hypotheses of the previous theorem hold. Suppose f has a unique best L_∞ approximation $F(A, \cdot)$ which cannot be expressed as $F(C, \cdot)$ with $Q(C, \cdot)$ having a zero in* [0, 1]. *Then* $\{F(A_k, \cdot)\}$ *converges uniformly to* $F(A, \cdot)$.

Convergence may not be uniform (or even pointwise) to a best L_∞ approximation even if the best L_∞ approximation is unique.

EXAMPLE. Let $f(x) = x - \frac{1}{2}$ and approximate by the family $R_1{}^0[0, 1]$ of ordinary rational functions. As $f - 0$ alternates once on [0, 1], 0 is the unique best L_∞ approximation to f. Let $F(A_k, \cdot)$ denote a best L_k approximation to f. By Theorem 2 of [7], $F(A_k, \cdot)$ cannot be zero for $k > 1$. Let $\{k(j)\}$ be a sequence such that $\{F(A_{k(j)}, \cdot)\}$ is of constant sign. Without loss of generality, we assume that this sign is positive. By Theorem 1 of [7], $f - F(A_{k(j)}, \cdot)$ has at least two sign changes on [0, 1]. As f and $F(A_{k(j)}, \cdot)$ are elements of $R_1{}^1[0, 1]$, $f - F(A_{k(j)}, \cdot)$ has at most two sign changes, hence it has exactly two sign changes. As $f(0) - F(A_{k(j)}, 0) < 0$ we have $f(1) - F(A_{k(j)}, 1) < 0$ and $F(A_{k(j)}, 1) > \frac{1}{2}$.

DEFINITION. σ is an *ordering function* if it is monotonic and strictly monotonic where it is finite.

DEFINITION. Associated with the parameter A is the linear space

$$S(A) = \{P(A, \cdot)\, Q(B, \cdot) - Q(A, \cdot)\, P(B, \cdot) : B \in E_{n+m}\}.$$

THEOREM 4. *Let the hypotheses of Theorem* 2 *hold. Let σ be an ordering function. Let $F(A, \cdot)$ be best to f, $Q(A, \cdot) > 0$, and $S(A)$ be a Haar subspace of dimension $n + m - 1$. Then $\{F(A_k, \cdot)\} \to F(A, \cdot)$ uniformly.*

Proof. σ being an ordering function and $S(A)$ being a Haar subspace implies that $F(A, \cdot)$ is a uniquely best admissible approximation [9]. Arguments similar to those of [8, middle of page 486] show that $F(A, \cdot)$ is uniquely best among approximations with denominators ≥ 0. $S(A)$ being of dimension $n + m - 1$ implies that $F(A, \cdot)$ has a unique representation. Apply Theorem 3.

COROLLARY. *Let the hypotheses of Theorem* 2 *hold. Let σ be an ordering function. Let the family of rationals be $R_m{}^n[0, 1]$. Let $F(A, \cdot)$ be best to f and $R(A, \cdot)$ be nondegenerate. Then $\{F(A_k, \cdot)\} \to F(A, \cdot)$ uniformly.*

The previous example shows that the hypotheses of dimension $n + m - 1$ or nondegeneracy cannot be dropped.

A best L_p approximation by admissible rationals may not exist. However, we have from Theorem 3

THEOREM 5. *Let the hypotheses of Theorem 2 hold. Suppose f has a unique best L_r approximation $F(A, \cdot)$ which cannot be expressed as $F(C, \cdot)$, $Q(C, \cdot)$ with a zero in* $[0, 1]$. *Then for all p sufficiently large, a best L_p approximation to f is admissible* (*hence a best admissible L_p approximation exists*).

REFERENCES

1. B. W. BOEHM, Existence of best rational Tchebycheff approximations, *Pacific J. Math.* **15** (1965), 19–27.
2. C. B. DUNHAM, Existence of best mean rational approximations, *J. Approximation Theory* **4** (1971), 269–273.
3. C. B. DUNHAM, Mean approximation by transformed and constrained rational functions, *J. Approximation Theory* **10** (1974), 93–100.
4. J. R. RICE, "The Approximation of Functions," Vol. 1, Addison–Wesley, Reading, Mass., 1964.
5. J. R. RICE, "The Approximation of Functions," Vol. 2, Addison–Wesley, Reading, Mass., 1969.
6. L. TORNHEIM, Approximation by families of functions, *Proc. Amer. Math. Soc.* **7** (1956), 641–643.
7. C. B. DUNHAM, Best mean rational approximation, *Computing (Arch. Elektron Rechnen)* **9** (1972), 87–93.
8. C. B. DUNHAM, Rational approximation on subsets, *J. Approximation Theory* **1** (1968), 484–487.
9. C. B. DUNHAM, Transformed rational Chebyshev approximation, *J. Approximation Theory* **19** (1977), 200–204.